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Type and Cotype in Lorentz L_{pq} Spaces

by J. Creekmore

Department of Mathematics, Kent State University, Kent, Ohio 44242, USA

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§ 1. INTRODUCTION

The past few years have evidenced a considerable increase in the degree of sophistication present in the study of the geometry of Banach spaces. The notions of type and cotype introduced by B. Maurey and G. Pisier are an example of this increased sophistication. The application of these and related notions to fundamental questions of Banach space structure theory bears witness to their importance. Perhaps nowhere has their effect been so obvious as in the study of the finer structure of Banach lattices.

In this note we gather a few of the tools developed by Maurey and Pisier and apply them to give a complete cataloguing of the type and cotype of the Lorentz function spaces L_{pq}. The reader will note that so powerful is this machinery of Maurey and Pisier that little work is needed to bring it to bear upon the case at hand. Therefore, were it not for the fact that the results herein appear to be among the first general linear topological invariants of the L_{pq} spaces discussed since Lorentz ([9], [10]), this note might well be viewed as an application of some powerful new techniques now available in the study of many function spaces.

§ 2. REVIEW OF BASIC DEFINITIONS AND RESULTS

The notions of type and cotype can be defined for arbitrary Banach spaces (see [8], 1.e.12).

DEFINITION 2.1. *A Banach space X is of type p for some $1 \leq p \leq 2$ if there exists a constant $M < \infty$ such that for every finite set $\{x_i\}_{i=1}^n$ in X,*

$$\int_0^1 \| \sum_{i=1}^n r_i(t) x_i \| dt \leq M (\sum_{i=1}^n \|x_i\|^p)^{1/p},$$

where $\{r_i\}_{i=1}^\infty$ denotes the Rademacher functions. X is of cotype q for some $2 \leq q < \infty$ if there exists a constant $M < \infty$ such that for every finite set $\{x_i\}_{i=1}^n$ in X,

$$(\sum_{i=1}^n \|x_i\|^q)^{1/q} \leq M \int_0^1 \| \sum_{i=1}^n r_i(t) x_i \| dt.$$

For example, as shown in [8], § 1.e, the Lebesgue spaces $L_p(\mu)$, $1 \leq p < \infty$, are of type min $(2, p)$ and of cotype max $(2, p)$, while c_0, the space of null sequences, is of no type $p > 1$ and of no cotype $q < \infty$. In order to study type and cotype in Banach lattices, we recall the related notions of p-convexity and q-concavity (see [8] 1.d.3).

DEFINITION 2.2. *Let $1 \leq p, q \leq \infty$. A Banach lattice X is said to be p-convex if there exists a constant $M < \infty$ such that for every finite set $\{x_i\}_{i=1}^n$ in X,*

$$\|(\sum_{i=1}^n |x_i|^p)^{1/p}\| \leq M (\sum_{i=1}^n \|x_i\|^p)^{1/p} \text{ if } 1 \leq p < \infty,$$

or $\|\sup_{1 \leq i \leq n} |x_i|\| \leq M \max_{1 \leq i \leq n} \|x_i\|$ if $p = \infty$. X is said to be q-concave if there exists a constant $M < \infty$ such that for every finite set $\{x_i\}_{i=1}^n$ in X,

$$(\sum_{i=1}^n \|x_i\|^q)^{1/q} \leq M \|(\sum_{i=1}^n |x_i|^q)^{1/q}\| \text{ if } 1 \leq q < \infty,$$

or $\max_{1 \leq i \leq n} \|x_i\| \leq M \|\sup_{1 \leq i \leq n} |x_i|\|$ if $q = \infty$.

Since for any functions $\{f_i\}_{i=1}^n$ in $L_p(\mu)$ we have

$$\|(\sum_{i=1}^n |f_i|^p)^{1/p}\|_p = (\sum_{i=1}^n \|f_i\|_p^p)^{1/p} \text{ if } 1 \leq p < \infty,$$

or

$$\|\sup_{1 \leq i \leq n} |f_i|\|_\infty = \max_{1 \leq i \leq n} \|f_i\|_\infty \text{ if } p = \infty,$$

$L_p(\mu)$ is both p-convex and p-concave. Thus the notions of p-convexity and p-concavity arise by replacing the above equalities by one-sided estimates.

REMARKS. 1. Note that every Banach lattice is 1-convex and ∞-concave.

2. For our purpose in studying type and cotype in L_{pq} spaces, it is enough for the Banach lattice to be a lattice of functions with the usual pointwise order. In general Banach lattices, the method of J. Krivine [6] is usually followed to give a precise meaning to the element $(\sum_{i=1}^n |x_i|^p)^{1/p}$ appearing in the above definition.

3. If in the definitions of type p and p-convexity only vectors $\{x_i\}_{i=1}^n$ with pairwise disjoint supports are considered, the definitions coincide. A similar situation holds in the definitions of cotype q and q-concavity. The notions obtained by considering only disjoint vectors is given in the following definition (see [8], 1.f.4).

DEFINITION 2.3. *Let $1<p,\ q<\infty$. A Banach lattice X is said to satisfy an upper p-estimate if there exists a constant $M<\infty$ such that for every finite set of pairwise disjoint vectors $\{x_i\}_{i=1}^n$ in X,*

$$\|\sum_{i=1}^n x_i\| \leq M(\sum_{i=1}^n \|x_i\|^p)^{1/p}.$$

X is said to satisfy a lower q-estimate if there exists a constant $M<\infty$ such that for every finite set of pairwise disjoint vectors $\{x_i\}_{i=1}^n$ in X,

$$(\sum_{i=1}^n \|x_i\|^q)^{1/q} \leq M\|\sum_{i=1}^n x_i\|.$$

We note that pairs of properties closely related to that given in Definition 2.3 have been defined in the literature. J.J. Grobler [4] introduced the l_p composition property and the l_p decomposition property for Banach function spaces, and P. Dodds [2] formulated these notions for Banach lattices. Dodds also introduced the strong l_p composition property and the strong l_p decomposition property, which by their definitions are identical to the respective notions of upper p-estimate and lower p-estimate. Moreover, Dodds has shown that a Banach lattice has the strong l_p decomposition property if and only if it has the l_p decomposition property, and W.K. Vietsch has shown in his thesis that a Banach lattice has the strong l_p composition property if and only if it has the l_p composition property (see [14], lemma 8.2, which is actually a special case of a result of P. Meyer-Nieberg). Thus, for Banach lattices, the above two pairs of properties coincide with that given in Definition 2.3.

Indices for Banach function spaces and Banach lattices were defined by Grobler and Dodds respectively. In both cases, the definitions are as follows:

$$s(X) = \sup\ \{p \geq 1\text{: the } l_p \text{ composition property holds in } X\}$$
$$\sigma(X) = \inf\ \{p \geq 1\text{: the } l_p \text{ decomposition property holds in } X\}.$$

In the course of our determination of the type and cotype of the L_{pq} spaces, we shall also compute the indices s_{pq} and σ_{pq} of these spaces. The following propositions, due to Maurey and Pisier, show some of the connections between the above three pairs of notions (see [8], § 1.d–§ 1.f).

PROPOSITION 2.4.

(i) *A q-concave Banach lattice, $q \geq 2$, is of cotype q.*

(ii) *A p-convex Banach lattice, $1 \leq p \leq 2$, which is also q-concave for some $q<\infty$, is of type p.*

The proofs follow easily from a generalization, due to Maurey, of the classical Khintchine inequality.

PROPOSITION 2.5. *Let X be a Banach lattice, $1<p<\infty$ and $1/p+1/p'=1$. Then*

(i) *X is p-convex (concave) if and only if its dual X^* is p'-concave (convex).*

(ii) *X satisfies an upper (lower) p-estimate if and only if its dual X^* satisfies a lower (upper) p'-estimate.*

It is obvious that Banach lattice satisfies an upper (lower) p-estimate provided it is p-convex (concave). The converse is false as will be seen in the next section. However, we do have the following result.

PROPOSITION 2.6. *If a Banach lattice satisfies an upper (lower) r-estimate for some* $1<r<\infty$, *then it is p-convex (q-concave) for every* $1<p<r<q<\infty$.

An immediate corollary is that an r-convex (concave) Banach lattice is p-convex (q-concave) for every $1<p<r<q<\infty$.

PROPOSITION 2.7. *A Banach lattice which satisfies a lower q-estimate,* $q>2$, *is of cotype q.*

PROPOSITION 2.8. *A Banach lattice is of type p,* $1<p\leq 2$, *if and only if its dual* X^* *is of cotype* p', $1/p+1/p'=1$, *and* X^* *satisfies an upper r-estimate for some* $r>1$.

§ 3. THE CATALOGUING OF THE TYPE AND COTYPE OF THE L_{pq} SPACES

We begin by recalling the definitions and basic properties of the L_{pq} spaces. Given a measure space (Ω, Σ, μ), where the measure μ is nonnegative and σ-finite, $L_{pq}(\mu)$ is the collection of (classes of) real-valued measurable functions f on Ω with $\|f\|^*_{pq}<\infty$, where

$$\|f\|^*_{pq}=\begin{cases}(q/p\int_0^\infty [t^{1/p}f^*(t)]^q\,dt/t)^{1/q}, & 1<p<\infty,\ 1\leq q<\infty\\ \sup_{t>0} t^{1/p}f^*(t), & 1<p<\infty,\ q=\infty.\end{cases}$$

Here, $f^*(t)$ is the nonincreasing rearrangement of f onto $(0, \infty)$ defined by $f^*(t)=\inf\{y>0: \lambda_f(y)\leq t\}$, where $\lambda_f(y)=\mu\{\omega\in\Omega: |f(\omega)|>y\}$ is the distribution function of f. If $q>p$, the weight function $t^{q/p-1}$ is increasing and $\|\cdot\|^*_{pq}$ fails the triangle inequality (see [10], p. 411). However, by considering the averaged function $f^{**}(t)=1/t\int_0^t f^*(s)ds$ and defining $\|f\|_{pq}=\|f^{**}\|^*_{pq}$, we obtain a norm for all values of p and q which is equivalent to $\|\cdot\|^*_{pq}$:

$$\|f\|^*_{pq}\leq\|f\|_{pq}\leq p/(p-1)\|f\|^*_{pq}.$$

As shown by R.A. Hunt in his survey paper on L_{pq} spaces [5], the dual space of L_{pq} for $q<\infty$ is $L_{p'q'}$, where $p^{-1}+(p')^{-1}=1=q^{-1}+(q')^{-1}$.

For convenience, we work with $\Omega=(0, \infty)$ and $\mu=$Lebesgue measure, although the proofs are easily modified to any nonnegative σ-finite measure space without atoms. Since the properties we consider are topological invariants, the equivalence of $\|\cdot\|^*_{pq}$ and $\|\cdot\|_{pq}$ will allow us to consider only $\|\cdot\|^*_{pq}$.

PROPOSITION 3.1. L_{pq} *is not p-concave for* $1\leq q<p$.

PROOF: Fix an integer m and let $\{\sigma_j\}_{j=1}^m$ map $\{1,2,\dots,m\}$ onto itself by $\sigma_j(i)=(i+j) \bmod m$. For $1\leq j\leq m$, put $f_j=\sum_{i=1}^m \chi_{(i-1,i]}/[\sigma_j(i)]^{1/p}]$. Then

$$\|(\textstyle\sum_{j=1}^m|f_j|^p)^{1/p}\|^*_{pq}=(\sum_{j=1}^m 1/j)^{1/p}\|\chi_{(0,m]}\|^*_{pq}\leq m^{1/p}(\log m+1)^{1/p}.$$

On the other hand, since $f_j^* = \sum_{i=1}^m \chi_{(i-1,i]}/i^{1/p}$ for all $1 \le j \le m$, we have

$$(\sum_{j=1}^m \|f_j\|_{pq}^{*p})^{1/p} = [m(\sum_{j=1}^m j^{q/p} - (j-1)^{q/p}/j^{q/p})^{p/q}]^{1/p} \ge cm^{1/p}(\log m)^{1/q}$$

for some $c>0$. Since m is arbitrary and $q<p$, this shows L_{pq} is not p-concave. By taking $f_j = \sum_{i=1}^m \chi_{(i-1/m,i/m]}/[\sigma_j(i)^{1/p}]$, the same proof shows $L_{pq}(0,1)$ is not p-concave for $1 \le q < p$.

In order to show L_{pq} does satisfy a lower p-estimate for $1 \le q \le p$, we make use of an alternate expression for $\|\cdot\|_{pq}^*$. Given a simple function $f = \sum_{i=1}^n c_i \chi_{E_i}$, where $\{E_i\}_{i=1}^n$ are pairwise disjoint measurable sets, let $c_{\sigma(1)} \ge c_{\sigma(2)} \ge \ldots \ge c_{\sigma(n)} \ge$ $\ge c_{\sigma(n+1)} = 0$ be the nonincreasing rearrangement of $\{|c_i|\}_{i=1}^n$ and let

$$d_i = \sum_{k=1}^i \mu(E_{\sigma(k)}),\ 1 \le i \le n \text{ with } d_0 = 0.$$

Then

$$\lambda_f(y) = \begin{cases} d_i, \text{ if } c_{\sigma(i+1)} \le y < c_{\sigma(i)},\ 1 \le i \le n \\ 0, \text{ if } y \ge c_{\sigma(1)} \end{cases}$$

so that

$$f^*(t) = \begin{cases} c_{\sigma(i)}, \text{ if } d_{i-1} \le t < d_i,\ 1 \le i \le n \\ 0, \text{ if } t \ge d_n \end{cases}.$$

Hence

$$\|f\|_{pq}^* = (q/p \int_0^\infty t^{q/p-1}(f^*(t))^q dt)^{1/q} = (\sum_{i=1}^n c_{\sigma(i)}^q (d_i^{q/p} - d_{i-1}^{q/p}))^{1/q},$$

while on the other hand

$$\begin{aligned}&(q \int_0^\infty y^{q-1}(\lambda_f(y))^{q/p} dy)^{1/q} = \\ &= (d_n^{q/p} c_{\sigma(n)}^q + d_{n-1}^{q/p}(c_{\sigma(n-1)}^q - c_{\sigma(n)}^q) + \ldots + d_1^{q/p}(c_{\sigma(1)}^q - c_{\sigma(2)}^q))^{1/q} = \\ &= (\sum_{i=1}^n c_{\sigma(i)}^q (d_i^{q/p} - d_{i-1}^{q/p}))^{1/q}.\end{aligned}$$

Since the simple functions are dense in L_{pq}, $q<\infty$ ([5]), we have

$$\|f\|_{pq}^* = (q \int_0^\infty y^{q-1}(\lambda_f(y))^{q/p} dy)^{1/q}.$$

PROPOSITION 3.2. *L_{pq} satisfies a lower p-estimate for $1 \le q < p$.*

PROOF: Let $f_1, f_2, \ldots, f_n \in L_{pq}$ with pairwise disjoint supports. Note that $\lambda_{\sum f_i} = \sum \lambda_{f_i}$. By the above observation, the desired inequality

$$(\sum_{i=1}^n \|f_i\|_{pq}^{*p})^{1/p} \le M \|\sum_{i=1}^n f_i\|_{pq}^*$$

becomes

$$\sum_{i=1}^n (q \int_0^\infty y^{q-1}(\lambda_{f_i}(y))^{q/p} dy)^{p/q} \le M^p (q \int_0^\infty y^{q-1}(\sum_{i=1}^n \lambda_{f_i}(y))^{q/p} dy)^{p/q},$$

i.e.,

$$\sum_{i=1}^n \|\lambda_{f_i}\|_{L_r(v)} \le M^p \|\sum_{i=1}^n \lambda_{f_i}\|_{L_r(v)},$$

where $r = q/p$ and $dv = qy^{q-1}dy$, which holds with $M=1$, since the quasi-norm in L_r is super-additive when $0<r<1$.

To show L_{pq} is q-convex for $1 \leq q < p$, we use another formula for $\|\cdot\|^*_{pq}$, valid only for $1 \leq q < p$. The formula states that

$$\|f\|^*_{pq} = \sup\, (q/p \int_0^\infty t^{q/p-1} |f(\tau(t))|^q dt)^{1/q},$$

where the supremum is taken over all measure-preserving automorphisms $\tau\colon \Omega \to \Omega$. Given any such automorphism τ, note that f and $f \circ \tau$ are equimeasurable (i.e., have the same distribution function), hence $f^* = (f \circ \tau)^*$. Since the inequality $\int_0^\infty f(t)g(t)d\mu(t) \leq \int_0^\infty f^*(t)g^*(t)dt$ holds for any measurable functions f and g (see [5], 1.9), we thus have

$$\|f\|^*_{pq} \geq \sup\, (q/p \int_0^\infty t^{q/p-1} |f(\tau(t))|^q dt)^{1/q}.$$

The reverse inequality follows by noting that given any two equimeasurable step functions f and g, there exists a measure-preserving automorphism $\tau\colon \Omega \to \Omega$ such that $f(t) = g(\tau(t))$.

PROPOSITION 3.3. *L_{pq} is q-convex for $1 \leq q < p$.*

PROOF. Given $f_1, f_2, \ldots, f_n \in L_{pq}$, we have

$$\|(\textstyle\sum_{i=1}^n |f_i|^q)^{1/q}\|^{*q}_{pq} = \sup\, (q/p \int_0^\infty t^{q/p-1} |(\sum_{i=1}^n |f_i|^q)^{1/q}(\tau(t))|^q dt)$$

$$= \sup\, (q/p \int_0^\infty t^{q/p-1} (\textstyle\sum_{i=1}^n |f_i(\tau(t))|^q) dt) \leq$$

$$\leq \textstyle\sum_{i=1}^n \sup\, (q/p \int_0^\infty t^{q/p-1} |f_i(\tau(t))|^q dt) = \sum_{i=1}^n \|f_i\|^{*q}_{pq}.$$

Using Proposition 2.5 and the fact that $L^*_{pq} = L_{p'q'}$, we can summarize as follows:

THEOREM 3.4. (a) *Let $1 \leq q < p$. Then*

(i) *L_{pq} is not p-concave, but satisfies a lower p-estimate.*
(ii) *L_{pq} is q-convex.*

(b) *Let $1 < p < q < \infty$. Then*

(i) *L_{pq} is not p-convex, but satisfies an upper p-estimate.*
(ii) *L_{pq} is q-concave.*

We are now prepared to exhibit the type and cotype of the L_{pq} spaces.

THEOREM 3.5. *Let $1 \leq q < p$. Then the following holds.*

(i) *L_{pq} is of type min $(2, q)$.*
(ii) *If $p \neq 2$, L_{pq} is of cotype max $(2, p)$.*
(iii) *L_{2q} is of cotype $(2 + \varepsilon)$, for all $\varepsilon > 0$.*

PROOF. (i) L_{pq} is min $(2, q)$-convex and satisfies a lower p-estimate, so by Proposition 2.6, L_{pq} is s-concave for any $p < s < \infty$. Hence, L_{pq} is of type min $(2, q)$ by Proposition 2.4(ii).

(ii) If $p < 2$, L_{pq} is 2-concave, hence of cotype 2 by Proposition 2.4(i). If $p > 2$, L_{pq} satisfies a lower p-estimate, so it is of cotype p by Proposition 2.4(i).

(iii) For any $\varepsilon>0$, L_{2q} is $(2+\varepsilon)$-concave and hence of cotype $(2+\varepsilon)$ by Proposition 2.4(i).

THEOREM 3.6. *Let $1<p<q<\infty$. Then the following holds.*

(i) *If $p\neq 2$, L_{pq} is of type min $(2, p)$.*

(ii) *L_{2q} is of type $(2-\varepsilon)$ for all $\varepsilon>0$.*

(iii) *L_{pq} is of cotype max $(2, q)$.*

PROOF. (i) If $p\neq 2$, $L^*_{pq}=L_{p'q'}$ $(1<q'<p'\neq 2)$ is of cotype max $(2, p')$ and satisfies an upper q'-estimate. By Proposition 2.8, L_{pq} is of type min $(2, p)$.

(ii) For any $\varepsilon>0$, L_{2q} is $(2-\varepsilon)$-convex and also q-concave, hence of type $(2-\varepsilon)$.

(iii) L_{pq} is max $(2, q)$-concave, so it is of cotype max $(2, q)$ by Proposition 2.4(i).

To complete our classification, we finally consider $L_{p\infty}$, $1<p<\infty$. As shown in [3], l_1 imbeds complementably into $L_{p,1}$, hence c_0 is isomorphic to a subspace of $L_{p\infty}$. Since c_0 is of no type $r>1$, and of no cotype $s<\infty$, we obtain

THEOREM 3.7. *Let $1<p<\infty$.*

(i) *$L_{p\infty}$ is of no type $r>1$.*

(ii) *$L_{p\infty}$ is of no cotype $s<\infty$.*

Using the results above, we can compute the indices s_{pq} and σ_{pq} of L_{pq} defined in § 2. Vietsch [14] has proved that $1\leq s_{pq}\leq \min(p,q)\leq \max(p,q)\leq \leq \sigma_{pq}\leq\infty$. We improve these estimates in our final result.

THEOREM 3.8. *Let s_{pq} and σ_{pq} be the indices of L_{pq} $(1<p<\infty,\ 1\leq q\leq\infty)$. Then $s_{pq}=\min(p,q)$ and $\sigma_{pq}=\max(p,q)$.*

PROOF. Since $L_{pp}=L_p(\mu)$, the case $p=q$ is trivial. Let $1\leq q<p$. By Theorem 3.4(a) and Proposition 2.6, $\sigma_{pq}=p$. Moreover, L_{pq} is q-convex so it satisfies an upper q-estimate. However, l_q imbeds into L_{pq} (see [3], p. 406), and by considering the unit vector basis of l_q it follows that l_q does not satisfy an upper r-estimate for any $r>q$. Hence $s_{pq}=q$.

The case $1<p<q\leq\infty$ follows by duality. Indeed, it follows from Proposition 2.5(ii) and the remarks after Definition 2.3 that if X is a Banach lattice with norm dual X^*, then $1/s(X)+1/\sigma(X^*)=1$ and $1/\sigma(X)+1/s(X^*)=1$. Using the fact that $L_{pq}=L^*_{p'q'}$, we obtain $s_{pq}=p$ and $\sigma_{pq}=q$.

§ 4. ADDED COMMENTS

We do not know if $(L_{pq}, \|\cdot\|_{pq})$ is uniformly convex, but we may address the question of uniform convexifiability in L_{pq} with use of results from T. Figiel and W.B. Johnson (see [8], 1.f.1). The conclusion to be drawn:

THEOREM 4.1. *Let $1<p,\ q<\infty$. Then*

(i) *L_{pq} is uniformly convexifiable with modulus of convexity of power type equal to the corresponding value of its cotype.*

(ii) *L_{pq} is uniformly smoothable with modulus of smoothness of power type equal to the corresponding value of its type.*

Another conclusion to be drawn from the results of § 3 concerns the linear topological structure of the $L_{p,1}$ spaces. As Banach lattices with finite cotype, $L_{p,1}$ does not contain c_0 and hence is weakly sequentially complete. In addition, $L_{p,1}$ is known to be a dual space [13], and being separable, has the Radon-Nikodym property [1]. However, $L_{p,1}$ is not an $\mathscr{L}_1$-space; indeed, by a theorem of D. Lewis and C. Stegall [7], an $\mathscr{L}_1$-space which is isomorphic to a subspace of a separable dual space imbeds into l_1, and it is easily seen that $L_{p,1}$ does not have the Schur property.

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